Exercise 1. (15 points)
We have the system

\[
A = \begin{pmatrix}
1 & -5 \\
7 & -1 \\
\end{pmatrix} \quad b = \begin{pmatrix}
-4 \\
6 \\
\end{pmatrix}
\]

We apply the Jacobi method

\[
x_1^{(k)} = -\frac{a_{12}}{a_{11}} x_2^{(k-1)} + \frac{b_1}{a_{11}} = 5 x_2^{(k-1)} - 4
\]

\[
x_2^{(k)} = -\frac{a_{21}}{a_{22}} x_1^{(k-1)} + \frac{b_2}{a_{22}} = 7 x_1^{(k-1)} - 6
\]

We take the given starting point \((x_1, x_2) = (0, 0)\) and perform 3 iterations:

\[
x_1^{(1)} = 5 \cdot 0 - 4 = -4
\]

\[
x_2^{(1)} = 7 \cdot 0 - 6 = -6
\]

\[
x_1^{(2)} = 5 \cdot (-6) - 4 = -34
\]

\[
x_2^{(2)} = 7 \cdot (-4) - 6 = -34
\]

\[
x_1^{(3)} = 5 \cdot (-34) - 4 = -174
\]

\[
x_2^{(3)} = 7 \cdot (-34) - 6 = -244
\]

The solution does not converge. If the matrix A was diagonally dominant, the method would converge. It is not the case here.

Exercise 2. (20 points)

a) Regula Falsi (10 points)

1st Iteration \((a_1 = 2.5, b_1 = \pi)\):
\[
c_i = b_i - \frac{f(b_i)(b_i - a_i)}{f(b_i) - f(a_i)} = 2.92793597
\]
\[
f(a_i) \cdot f(c_i) = 32.48 > 0, \text{ so } a_2 = c_1
\]

\(2^{nd} \text{ Iteration} \ (a_2 = 2.92793597, b_2 = \pi):\)
\[
c_2 = b_2 - \frac{f(b_2)(b_2 - a_2)}{f(b_2) - f(a_2)} = 3.00645312
\]
\[
f(a_2) \cdot f(c_2) = 1.331 > 0, \text{ so } a_3 = c_2
\]

\(3^{rd} \text{ Iteration} \ (a_3 = 3.00645312, b_3 = \pi):\)
\[
c_3 = b_3 - \frac{f(b_3)(b_3 - a_3)}{f(b_3) - f(a_3)} = 3.01671883
\]

**Accuracy of solution:**
\[
|x^* - c_3| = 0.00144538
\]

\[b) \text{ Newton Raphson (10 points)}\]

**Derivative:**
\[
f'(x) = 2 \cdot e^x (\sin(x) + \cos(x)) - 2
\]

**Starting point:**
\[
x_0 = 2.5
\]

\(1^{st} \text{ Iteration:}\)
\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.02517042
\]

\(2^{nd} \text{ Iteration:}\)
\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.43863015
\]

**Accuracy of solution:**
\[
|x^* - x_2| = 0.42046594
\]

**Exercise 3. (25 points)**

\[a) \text{ Adams-Bashforth-Moulton (15 points)}\]

1) Since the Adams-Bashforth-Moulton method is not a self-starting method, we need to calculate 3 initial points:
\[
\frac{dx}{dt} = -2x(t)
\]
\[
h = \frac{1}{3}
\]
\[
x(t_0) = 3
\]
\[x(t_1) = x(t_0) + h \cdot f(x(t_0), t_0) = 3 - \frac{1}{3} \cdot 6 = 1
\]
\[x(t_2) = x(t_1) + h \cdot f(x(t_1), t_1) = 1 - \frac{1}{3} \cdot 2 = \frac{1}{3}
\]
\( x(t_3) = x(t_2) + h \cdot f(x(t_2), t_2) = \frac{1}{3} - \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{9} \)

2) Application of the method

\[
\begin{align*}
x(t_4) &= \frac{1}{9} + \frac{1}{72} \cdot \left( 55 \cdot \left(-\frac{2}{9}\right) + 59 \cdot \left(\frac{2}{3}\right) - 37 \cdot 2 + 9 \cdot 6 \right) = 0.2 \\
x(t_5) &= \frac{1}{9} + \frac{1}{72} \cdot \left( 9 \cdot (-0.4) + 19 \cdot \left(-\frac{2}{9}\right) - 5 \cdot \left(-\frac{2}{3}\right) - 2 \right) = 0.02 \\
x'(t_5) &= 0.02 + \frac{1}{72} \cdot \left( 55 \cdot (-0.04) + 59 \cdot \left(\frac{2}{9}\right) - 37 \cdot \left(-\frac{2}{3}\right) + 18 \right) = 0.07 \\
x(t_5) &= 0.02 + \frac{1}{72} \cdot \left( 9 \cdot (-0.16) - 19 \cdot 0.04 + 5 \cdot \left(\frac{2}{9}\right) - \frac{2}{3} \right) = 0
\end{align*}
\]

b) (5 points)
The Adam-Bashforth-Moulton method can be used to solve stiff systems. This is because the Adams-Moulton corrector is an implicit method.

c) (5 points)
The Euler backward method is an implicit method. This is because in order to estimate the point \( x(t_{k+1}) \) the algorithm require the value of \( f(x(t_{k+1}), t_{k+1}) \).

**Exercise 4. (15 points)**

a) Central differences (6 points)

At \( t=1 \) s, we can use \( h=2 \) s, therefore we get

\[
a'(t_0) = \frac{v(t_0 + \frac{h}{2}) - v(t_0 - \frac{h}{2})}{h} \quad \Rightarrow \quad a(1) = \frac{v(2) - v(0)}{2} = 12.4 \text{ m/s}^2
\]

At \( t=4 \) s, we have to take \( h=4 \) s, therefore we get

\[
a(4) = \frac{v(6) - v(2)}{4} = 5.7 \text{ m/s}^2
\]

The accuracy is better at \( t=1 \) s as we have used the smaller differentiation step.

b) Integration (9 points)

Since the data are given in 5 equidistant points (i.e. 4 intervals) we can use Simpson 1/3. Therefore, we have

\[
l = \frac{h}{3} \left[ f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + f(t_4) \right] = 280.333 \text{ m}
\]

**Exercise 5. (25 points)**

a) (3 points)
The missing code at * is:

\[
terrF = \text{norm}(Ax - b);
\]

b) (5 points)
The mistake in the code: for negative values of \( \text{Vin} \) elements the size of the vector \( \text{Vin} \) is reduced in the loop causing the problems with accessing all the elements of \( \text{Vin} \). One of the possible solutions is to store the positive elements in a new vector. Another is to access the elements of \( \text{Vin} \) in the decreasing order of indices.

c) (8 points)
The missing code:

```matlab
while (diff_x>convThresh) && (iterationN<max_it)
    x_new = x_old - fun(x_old)./dfun(x_old);
    ...
    iterationN = iterationN + 1;
end
```

d) (3 points)

```matlab
N = 10;
r = 0.5;
V = r .^ (0:N); % another option would be V = power(r,0:N);
```

e) (3 points)

The function computes the derivative of a function \( fH \) at \( x_0 \) using the central differences method;

f) (3 points)

```matlab
x0 = 2;
h = 1e-6;
fH = @(x) (sin(x)*exp(2*x));
[F, dF] = myFunction(fH, x0, h);
```