**Problem1**

**a.** We use LU decomposition method to solve the system of linear equations:

$$A = \begin{matrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{matrix} \quad = \quad LU$$

We need to find L and U after decomposing the Matrix A. Where,

$$L = \begin{matrix} l11 & 0 & 0 \\ l21 & l22 & 0 \\ l31 & l32 & l33 \end{matrix} \quad \text{and} \quad U = \begin{matrix} 1 & u12 & u13 \\ 0 & 1 & u23 \\ 0 & 0 & 1 \end{matrix}$$

with

$$LU = \begin{matrix} l11 & l11.u12 & l11.u13 \\ l21 & l21.u12 + l22 & l21.u13 + l22.u23 \\ l31 & l31.u12 + l32 & l31.u13 + l32.u23 + l33 \end{matrix}$$

The entries of LU matrix found by comparing A and LU:
l11=1
l21=3
l31=2
u12=2/1=2
l22=8-3*2=2
l32=6-2*2=2
u13=4/1=4;
u23=(14-(3*4))/2=1
u33=13-(2*4)-(2*1)=3

Now we have L and U matrices

$$L = \begin{matrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 3 \end{matrix} \qquad U = \begin{matrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$$

We can apply back substitution method

AX=B
LUX=B
UX=y
LY=B

**b.** <u>Brent:</u> B= $\begin{matrix} 4 \\ 14 \\ 12 \end{matrix}$

we are solving LY=B

$$\begin{matrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 3 \end{matrix} \quad \begin{matrix} y1 \\ y2 \\ y3 \end{matrix} = \begin{matrix} 4 \\ 14 \\ 12 \end{matrix}$$

y1=4
3y1+2y2=14
2y1+2y2+3y3=12

We get y1=4 y2 =1 y3=2/3

Now, we solve (UX=Y)

$$\begin{matrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x1 \\ x2 \\ x3 \end{matrix} = \begin{matrix} 4 \\ 1 \\ 2/3 \end{matrix}$$

Final solutions for x:
x3=2/3
x2=1/3
x1=4-(2*1/3)-4*2/3=2/3;

Similarly, for <u>Abadan:</u> B= $\begin{matrix} 3 \\ 11 \\ 9 \end{matrix}$

$$\begin{matrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 3 \end{matrix} \quad \begin{matrix} y1 \\ y2 \\ y3 \end{matrix} = \begin{matrix} 3 \\ 11 \\ 9 \end{matrix}$$

y1=3
3y1+2y2=11
2y1+2y2+3y3=9

y1=3 y2 =1 y3=1/3

$$\begin{matrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x1 \\ x2 \\ x3 \end{matrix} = \begin{matrix} 3 \\ 1 \\ 1/3 \end{matrix}$$

Final solution for (b):

x3=1/3
x2=2/3
x1=3-(2*2/3)-4*1/3=1/3;

## Problem 2

a) Use forward differences for the first point, backward differences for the last point and central differences everywhere else.

Forward differences:

$$\frac{dh_1}{dt} = \frac{h_2 - h_1}{\Delta t}$$

Backward differences:

$$\frac{dh_{11}}{dt} = \frac{h_{11} - h_{10}}{\Delta t}$$

Central differences:

$$\frac{dh_i}{dt} = \frac{h_{i+1} - h_{i-1}}{2\Delta t}$$

| Time (s) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Level (m) | 1.000 | 1.220 | 1.232 | 1.059 | 0.937 | 0.872 | 0.841 | 0.827 | 0.821 | 0.818 | 0.817 |
| dh/dt (m/s) | 2.20E-2 | 1.16E-2 | -8.04E-3 | -1.48E-2 | -9.35E-3 | -4.76E-3 | -2.25E-3 | -1.03E-3 | -4.69E-4 | -2.12E-4 | -1.32E-4 |

b) Compute $F_{in}(t)$ as $F_{in}(t) = A\frac{dh(t)}{dt} + S_n\sqrt{2gh(t)}$

| Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Time (s) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| $F_{in}(t)$ ($m^3/s$) | 2.5535E-4 | 2.6310E-4 | 2.3350E-4 | 2.0481E-4 | 2.0052E-4 | 1.9911E-4 | 2.0002E-4 | 1.9987E-4 | 2.0067E-4 | 2.0031E-4 | 2.0018E-4 |

a) To compute the overall volume of the liquid that was pumped into the reactor we integrate the volumetric flow rate $F_{in}(t)$ over the observed time using a composite 1/3 Simpson rule for 10 intervals as we have 11 points.

$$V = \int_{t=0}^{t=100} F_{in}(t)\, dt$$

$$V \approx \frac{(t_{10}-t_0)}{3n}\left(F_{in}(t_0)+4\sum_{i=1,3,5,....}^{n-1} F_{in}(t_i)+2\sum_{i=2,4,6,....}^{n-2} F_{in}(t_i)+F_{in}(t_2)\right)$$

$$V \approx \frac{100s \cdot 10^{-4}}{3 \cdot 10} \cdot (2.5535 + 4 \cdot (2.6310 + 2.3350 + 1.9911 + 1.9987 + 2.0031) + 2 \cdot (2.3350 + 2.0052 + 2.0002 + 2.0067) + 2.0018) \frac{m^3}{s}$$

$$V \approx 0.0217\, m^3 = 21.7\, L$$

**Problem 3**

a) Euler backward method:

$$\frac{G(t_{k+1})-G(t_k)}{h} = f_1(G(t_{k+1}),t_{k+1}) + O(h)$$
$$\frac{L(t_{k+1})-L(t_k)}{h} = f_2(L(t_{k+1}),t_{k+1}) + O(h)$$

neglecting $O(h)$ and replacing $f_1$ and $f_2$ we will have:

$$G(t_{k+1}) - G(t_k) - h\big(0.7G(t_{k+1}) - 0.007G(t_{k+1})L(t_{k+1})\big) = 0$$
$$L(t_{k+1}) - L(t_k) - h\big(0.0021G(t_{k+1})L(t_{k+1}) - 0.5L(t_{k+1})\big) = 0$$

taking $w_1 = G(t_{k+1})$ and $w_2 = L(t_{k+1})$ and replacing $G(t_k)$ and $L(t_k)$ by $G(0)$ and $L(0)$ we will have :

$$F_1 = w_1 - 400 - h(0.7w_1 - 0.007w_1w_2) = 0$$
$$F_2 = w_2 - 50 - h(0.0021w_1w_2 - 0.5w_2) = 0$$

Newton-Raphson can be used for solving this multi-dimensional root finding problem. Therefore, in each iteration we have:

$$w_{k+1} = w_k - J^{-1}(w_k)F(w_k)$$

where for the first iteration:

$$w_k = \begin{bmatrix} G(0) \\ L(0) \end{bmatrix} = \begin{bmatrix} 400 \\ 50 \end{bmatrix} \text{ and } J(w_k) = \begin{bmatrix} \frac{\partial F_1}{\partial w_1} & \frac{\partial F_1}{\partial w_2} \\ \frac{\partial F_2}{\partial w_1} & \frac{\partial F_2}{\partial w_2} \end{bmatrix} =$$
$$\begin{bmatrix} 1 - h \cdot 0.7 + 0.007 \cdot 50 \cdot h & 0.007 \cdot h \cdot 400 \\ -h \cdot 0.0021 \cdot 50 & 1 - 0.0021 \cdot 400 \cdot h + 0.5 \cdot h \end{bmatrix}$$

$$J^{-1}(w_k) = \frac{1}{\frac{\partial F_1}{\partial w_1}\cdot\frac{\partial F_2}{\partial w_2} - \frac{\partial F_1}{\partial w_2}\cdot\frac{\partial F_2}{\partial w_1}} \begin{bmatrix} 1 - 0.0021 \cdot 400 \cdot h + 0.5 \cdot h & -0.007 \cdot h \cdot 400 \\ h \cdot 0.0021 \cdot 50 & 1 - h \cdot 0.7 + 0.007 \cdot 50 \cdot h \end{bmatrix}$$

$$F(w_k) = \begin{bmatrix} 400 - 400 - h(0.7 \cdot 400 - 0.007 \cdot 400 \cdot 50) \\ 50 - 50 - h(0.0021 \cdot 400 \cdot 50 - 0.5 \cdot 50) \end{bmatrix}$$

Therefore, the values of $x$ at each time step are calculated using value of $x$ at previous time step and sufficient iterations of Newton-Raphson.

b) For each iteration we have:

$$w_{k+1} = w_k - J^{-1}(w_k)F(w_k)$$

using $h = 1/104$ we have
First iteration:

$$\begin{bmatrix} w_1(1) \\ w_2(1) \end{bmatrix} = \begin{bmatrix} 400 \\ 50 \end{bmatrix} -= \begin{bmatrix} 400 \\ 50 \end{bmatrix} - \begin{bmatrix} 1.003 & -0.027 \\ 0.001 & 1.003 \end{bmatrix} \begin{bmatrix} -1.346 \\ -0.163 \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} w_1(2) \\ w_2(2) \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix} - \begin{bmatrix} 1.003 & -0.027 \\ 0.001 & 1.003 \end{bmatrix} \begin{bmatrix} 1.498 * 10^{-5} \\ -4.495 * 10^{-6} \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix}$$

Third iteration:

$$\begin{bmatrix} w_1(3) \\ w_2(3) \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix} - \begin{bmatrix} 1.003 & -0.027 \\ 0.001 & 1.003 \end{bmatrix} \begin{bmatrix} 8.882 * 10^{-15} \\ 1.776 * 10^{-15} \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix}$$

therefore, the estimation for $x(1)$ is:

$$x(1) = \begin{bmatrix} G(1) \\ L(1) \end{bmatrix} = \begin{bmatrix} 401.346 \\ 50.165 \end{bmatrix}$$

c) Explicit methods are applicable for non-stiff systems and Runge-Kutta (4,5) (ODE45 in MatLab) is one of the best explicit methods. Implicit methods such as Adams-Bashforth-Moulton predictor-corrector are applied when we have a stiff system, since they are more robust.

**Problem 4**

Solution a)

```
i = 1;
while A(i) ~= 0 && i <= length(A)
    i = i + 1 ;
end
iZero = i;
```

Solution b)

```
iZero = min(find(A==0));
```

## Problem 5

Solution a)

```
tspan = [0,10];
x0 = [1,0]
[t,x] = ode45(f,tspan,x0);
```

Solution b)

Variant 1: Anonymous function

```
f = @(t,x)[x(1)*t – 2*x(2); 10*x(1) + 5]
```

Variant 2 – As a matlab function

```
function dxdt = f(x,t)
      dxdt =  [x(1)*t - 2*x(2); 10*x(1) + 5];
end
```

## Problem 6

1. c.
2. b.
3. e.
4. a.
5. d.

## Problem 7

a) `function [A,B] = functionExam(x,y,z)`

b) `A = functionExam(1,3)`

c) `[A,B] = functionExam(2,3,5)`